# Math 206A Lecture 19 Notes

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## 1 Bar and Joint Frameworks and Rigidity

### 1.1 Frameworks and static rigidity

**Definition 1.1.** A **bar and joint framework** is a pair $(G, L)$ where $G = (V, E)$ is a graph, and $L$ is a length function such that $L(e)$ is the length of $e_{i,j}$.

**Definition 1.2.** A **realization** of a bar and joint framework is a map $f : V \to \mathbb{R}^d$ such that for all edges $e = (i, j)$, $|f(i) - f(j)| = L(e)$.

This gives us a new interpretation of Cauchy's theorem.

**Theorem 1.1.** *If $P \subseteq \mathbb{R}^3$ is a simplicial polytope and $(G, L)$ is a corresponding framework, then there exists a unique convex realization of $(G, L)$.*

Here is a corollary of the 4-color theorem.

**Theorem 1.2.** *Suppose $G$ is a planar graph, and $L(e) = 1$ for all $e$. Then there exists a realization $f : G \to \mathbb{R}^3$.*

**Definition 1.3.** A realization is **statically rigid** if there does not exist a nonzero function $\lambda : E \to \mathbb{R}$ such that for every $v \in V$, $\sum_{(v,w)\in E} \lambda((v, w)) \cdot (vw) = 0$, where $vw$ denotes the vector from $v$ to $w$ in $\mathbb{R}^3$.

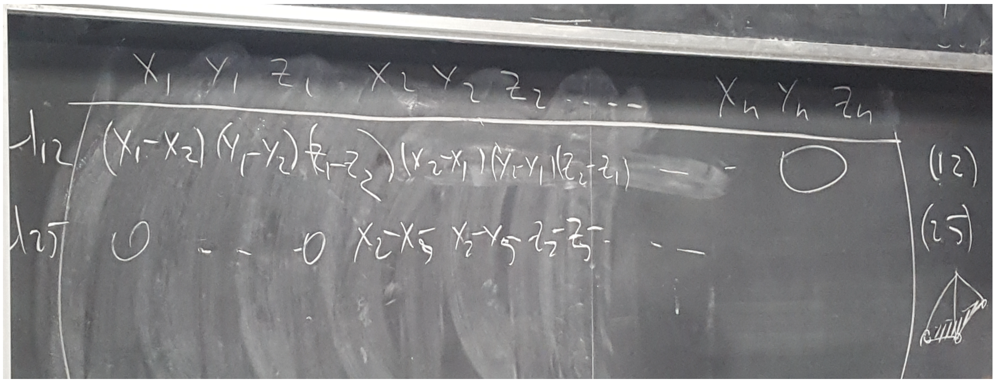
The function $\lambda$ basically allows us to change the length slightly to have a little flexibility in our realizations.

### 1.2 Dehn's rigidity theorem

**Theorem 1.3** (Dehn)**.** *Let $(G, L)$ be a framework of a simplicial polytope in $\mathbb{R}^3$. Then it is statically rigid.*

B. Fuller came up with an architectural design for a dome which is statically rigid. It needs to pillars to stand up. You can prove that it is statically rigid using Dehn's theorem.

**Definition 1.4.** Let each vertex be $v_i = (x_i, y_i, z_i)$. Construct a $3n \times 3n - 6$ matrix as follows:




The **rigidity matrix** $R_G$ is the $(3n-6) \times (3n-6)$ where we delete 6 of the columns.

Static rigidity means that the rigidity matrix $R_G$ has full rank. The idea of deleting the 6 columns is that we are grounding a triangular face of the polytope.

**Lemma 1.1.** *Dehn's theorem is equivalent to* $\det(R_G) = 0$.

*Proof.* Here is the idea. The determinant of $R_G$ is the sum of terms times $(-1)$ to the something. We show that there is at least 1 nonzero term, and then we show that all terms have the same sign. $\square$